SPONTANEOUS COORDINATION

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ABSTRACT

In this paper we study large potential games with global random matching. The model defines a discrete stationary Markov process, in discrete time. We derive the limiting distribution for the paradigmatic case of coordination 2x2 games.We show that spontaneous coordination is possible for substantial noise levels. Whether coordination occurs depends on the product of the incentives (benefit of coordinating, J) and an incentives responsiveness parameter that captures noise in the system (β). If the product is below a critical level, players fail to coordinate; if it is above, the distribution will be bi-modal with the highest peak close to the risk-dominant strategy.

Keywords: Markov process, Limiting distribution, Adaptive model, Coordination, Noisy best response

INTRODUCTION

Theoretical economics has recently begun to explore the implications of models of bounded rationality and learning.[2] The basic idea is as old as Nash equilibrium. In his unpublished Ph.D. dissertation, John Nash wrote the following:[3]

> It is unnecessary to assume that the participants have full knowledge of the total structure of the game, or the ability and inclination to go through any complex reasoning processes. But the participants are supposed to accumulate empirical information on the relative advantages of the various pure strategies at their disposal.
> To be more detailed, we assume that there is a population (in the sense of statistics) of participants for each position of the game. Let us also assume that the 'average playing' of the game involves n participants selected at random from the n populations, and that there is a stable average frequency with which each pure strategy is employed by the 'average member' of the appropriate population (...) Thus the assumption we made in this 'mass-action' interpretation leads to the conclusion that the mixed strategies representing the average behavior in each of the populations form an equilibrium point.

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[2] For overviews see e.g. Fudenberg and Levine (1998), Young (1998), Blume (1997).

[3] Nash, pp. 21-23, quoted in Weibull (1997).

Most of the work, however, has focused on providing behavioral foundations for existing solution concepts, most importantly Nash-Equilibrium and its refinements (Foster and Young (1990), Blume (1993), Blume (1995), Kandori, Mailath, and Rob (1993), Young (1993)). The general strategy of this approach can be described as follows. For a given normal form game, the researcher defines some version of perturbed best response that induces a regular Markov process. Since regular Markov processes yield unique stationary distributions, it is then shown that for vanishing noise in the limit, the unique stationary distribution puts positive probability only on a subset of possible action configurations, the so-called "stochastically stable states" (Foster and Young (1990)). This approach thus suggests answers to two important questions in classical game theory: the problem of coordination and the problem of selection. Ordered states emerge without a central coordinating device and are frequently unique even in the case of many strict Nash equilibria.

In this approach the use of behavioral models is largely foundational. Consequently, issues of robustness, e.g. with respect to distributional assumptions and so forth, are of central concern (Foster and Young (1990), Bergin and Lipman (1996)). In this paper we take a different approach. Rather than using the behavioral models in a foundational sense we use them in an explanatory way. In other words, we will directly explore the properties of limiting distributions rather than just exploring the limits as the noise term vanishes. The goal is to explore the properties of a certain behavioral approach directly and assess whether it can provide additional insights.

The basic model can be described as follows. In each period one agent is randomly selected to change his behavior.[4] That agent then will receive (possibly partial) information about the current state of play, i.e. the vector of actions currently chosen by each agent. Based on this information the agent responds by choosing an action, according to some (noisy) best-response rule. That is, with some probability the agent chooses a best response against the current configuration of play, but with some probability he chooses some other action. The realization of that action then determines the next period's configuration of play; again an agent is chosen (with replacement) and so forth. The key idea of our model is to "decompose" the simultaneous choice of classical game-theory (where agents form conjectures about each others beliefs) into a dynamic adjustment process.

As in classical game-theory, some features of the approach model are mainly technical, while others are of substantive importance. One of the technical assumptions pertains to selecting exactly one agent in each period. This does not imply that agents cannot change their behavior "quickly". After all, periods between revisions can be arbitrarily small. The *informational* implication of this assumption, however, is substantial. That is, when revising their actions, agents base their actions on their information about the current state of the dynamic system. This information may be complete (i.e. agents observe the complete current state of play), partial (i.e. they may only receive information about some agents, i.e. their "neighbors" on some network structure), or noisy (i.e. information may come in the form of polls). How this informational structure is modeled may have important consequences on the behavior of the system.

Among the substantive assumptions, perhaps the most important pertains to bounded

[4]We focus here on a discrete time process. Equivalently, we could consider continuous time formulations where the time between revisions is exponentially distributed.

rationality. First, agents are assumed to be myopic. They optimize conditional on the current behavior in the population without anticipating the future strategic consequences of their actions. Agents are not assumed to believe that other actors reason in the same way as they do, or even that they have the same payoff function. Indeed, they do not expect that their action may influence the future decisions of other participants. Agents simply base their choice on what action maximizes their current payoff.

Second, agents are assumed to respond to incentives, but not perfectly. That is, agents behavior is characterized by noisy decision making. While the main motivation for this assumption is technical[5], it has a plausible foundation in the random utility model due to McFadden (1973).[6] This approach seems especially appropriate in models of decision situations where the perceived costs and benefits may vary over time. Finally, we are mainly interested in applications to mass behavior such as collective action problems or conventions. We will therfore focus on large-N approximations of the induced Markov process.

In this paper we derive the closed form solution for the paradigmatic case of coordination 2×2 games[7]. We also show that important phenomena are missed if attention is exclusively focused on the case of vanishing noise. First, spontaneous coordination is possible for substantial noise levels. Second, whether coordination occurs depends on the product of the incentives generated by the payoff matrix and an incentive responsiveness parameter that captures the noise in the system. If the product is below a critical level, players will not coordinate. Rather, the limiting distribution will be unimodal. If this product is above a critical level, the stationary distribution will be bimodal with both peaks close to the two Nash-equilibria and the higher peak associated with the risk-dominant equilibrium. Thus, everything else equal, higher payoff differences would result in higher expected levels of coordination.

GENERAL MODEL

We consider a large population of players $\Im$ (with $|\Im| = KN$) that is partitioned into K types. Players interact according to a K-type strategic form game. Players interact repeatedly with each other according to a model of adaptive play with persistent randomness. At each point in time each player's decision gives rise to the population's configuration of play. The set of these configurations C then corresponds to a state space of a stochastic process induced by individual adaptive behavior.

Let S_k be the (finite) number of pure strategies $s_l^k (l = 1, ..., S_k)$ for a player of type k. Further let $n(s_l^k)$ be the number of players of type k playing strategy s_l^k. Then

$$\mathbf{n} := (n(s_1^1), ..., n(s_{S_1}^1), ..., n(s_l^k), ..., n(s_1^K), ..., n(s_{S_K}^K))$$

[5]Together with random selection it ensures that the stochastic process is ergodic.

[6]Alternatively, agents may be assumed to make a "mistake" with a fixed probability. See Kandori, Mailath and Rob (1993) or Young (1993) for details.

[7]The game is widely used in political science applications, for example in the study of collective action (e.g. Chong (1991)).

is a generic configuration of the set C. We allow only for elementary adjustments. That is, at most one individual of type k can change her behavior at a given time from s_l^k to $s_{l'}^k$. Thus from a given state n only states of the form

$$\mathbf{n}' := (n(s_1^1), ..., n(s_{S_1}^1), ..., n(s_l^k) - 1, ..., n(s_{l'}^k) + 1, ...n(s_1^K), ..., n(s_{S_K}^K))$$

are accessible.

Transition probabilities $P(\mathbf{nn}')$ are defined by selection and action probabilities. Selection probabilities are defined as follows: in each period t one specific agent out of KN is randomly chosen with probability $1/(KN)$.[8] The agent then looks at the current configuration $\mathbf{n}$ of actions in the population and adjusts his action according to a given behavioral rule. During the next period, again a player (perhaps the same) is chosen at random, and so on. Given the current configuration, an actor will then probabilistically adjust his participation behavior to improve his payoff.

Let $p^\beta(s_l^k|\mathbf{n})$ denote the conditional probability that in the next period an agent will play strategy s_l^k given that the current configuration of play is $\mathbf{n}$. Specifically, we assume log-logistic response rules for all agents:

$$p^\beta(s_l^k|\mathbf{n}) = \frac{\exp[\beta u(s_l^k; \mathbf{n})]}{\sum\limits_{s_{l'}^k} \exp[\beta u(s_{l'}^k; \mathbf{n})]}.$$

This captures the assumption that the pair-wise probability ratios of choosing actions are proportional to the respective pay-off differences. The log-linear choice model is closely connected to the best-response correspondence. The parameter β formally captures the degree to which the deterministic component of utility (given by the payoff matrix) determines choice. A low β corresponds to the case where a decision is not much influenced by the incentives specified in the model. For $\beta = 0$ choice is completely random. That is, for all possible configurations, the agent will play each action with equal probability. For $\beta \to \infty$, log-linear choice converges to a distribution that puts positive probability only on best-responses to $\mathbf{n}$.

Action and selection probabilities define a (regular) Markov chain with a unique limiting distribution π that satisfies the global balance conditions $\pi = \pi P$. In general, global balance equations cannot be solved in closed form. However, for the case where a k-player game has a potential $F(\mathbf{n})$ (Monderer and Shapley (1996)), Blume (1997) has shown that the limiting distribitution satisfies a stronger property, the *detailed balance condition.*

$$P(\mathbf{nn}')\pi(\mathbf{n}) = P(\mathbf{n}'\mathbf{n})\pi(\mathbf{n}')$$

We then immediately have the following result:

Theorem 1 *Let $\{\mathbf{n}[1], ..., \mathbf{n}[j\}, ...\mathbf{n}[m]\}$ be a sequence of configurations of length m. For log-logistic adjustment rule in games with a potential stationary distributions are of the form*

$$\pi(\mathbf{n}[m]) = \prod_{j=1}^{m-1} \frac{P(\mathbf{n}[j], \mathbf{n}[j+1])}{P(\mathbf{n}[j+1], \mathbf{n}[j])} \pi(\mathbf{n}[1]).$$

[8]For simplicity, we assume that revisions are made each period. All results, however, continue to hold in continuous time when the time between revisions is exponentially distributed.

2x2 COORDINATION GAMES WITH RANDOM GLOBAL MATCHING

We now apply this approach to the paradigmatic case of coordination problems: 2×2 coordination game with random global matching. So, consider a symmetric 2×2 games with payoff matrix

$$M = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}.$$

We denote the top/left action by s_1, the bottom/right action by s_2. Then, for random global matching, a player i's total payoff from a configuration $\mathbf{n}$ is given as

$$u_i(\mathbf{n}) := \sum_{i' \in \Im \setminus \{i\}} \frac{1}{2N-1} u_{ii'}$$

It can be easily seen that the following matrix F constitutes a potential function for the game M.

$$F = \begin{pmatrix} \frac{3u_{11}+u_{12}-3u_{21}-u_{22}}{4} & \frac{-u_{11}+u_{12}+u_{21}-u_{22}}{4} \\ \frac{-u_{11}+u_{12}+u_{21}-u_{22}}{4} & \frac{-u_{11}-3u_{12}+u_{21}+3u_{22}}{4} \end{pmatrix}$$

By defining

$$J = \frac{u_{11} - u_{12} - u_{21} + u_{22}}{2} \text{ and } h = \frac{u_{11} + u_{12} - u_{21} - u_{22}}{4}$$

we can rewrite F as[9]

$$F = \begin{pmatrix} \frac{J}{2} + 2h & -\frac{J}{2} \\ -\frac{J}{2} & \frac{J}{2} - 2h \end{pmatrix}$$

Further, if $F(\cdot)$ exists and is symmetric for a 2×2 game we can easily define a potential function $F(\cdot)$ for any finite network game (Blume (1997)). Thus, we can apply Theorem 1 to derive the limiting distribution.

Since, $\mathbf{n} = (n(s_1), n(s_2))$, each configuration is uniquely determined by $\frac{n(s_1)-n(s_2)}{2}$ which we abbreviate by n. By a slight abuse of notation, we also refer to each state by n, and write $n+1$ for $\mathbf{n}' = (n(s_1)+1, n(s_2)-1)$ and $n-1$ if $\mathbf{n}' = (n(s_1)-1, n(s_2)+1)$. Let $p_s(n)$ denote the (selection) probability of chosing a player using action $s = s_1, s_2$ in state n. Then

$$p_{s_1}(n) = \frac{n(s_1)}{2N} = \frac{N+n}{2N} \qquad \text{and} \qquad p_{s_2}(n) = \frac{n(s_2)}{2N} = \frac{2N - n(s_1)}{2N} = \frac{N-n}{2N}$$

[9] In coordination games, i.e. $u_{11} > u_{21}, u_{22} > u_{12}, u_{11} \geq u_{22}$, these parameters have a natural interpretation. The parameter $+J$ measures the benefit from coordinating. The parameter h indicates which of the two equilibria is risk-dominant. Since in coordination games we have $J > 0$, the argmax set of $F(\cdot)$ depends on h. If $h > 0$, then (s_1, s_1) is risk-dominant. If $h < 0$, (s_2, s_2) is risk-dominant

We will now apply Theorem 1 to explicitly calculate $\pi(n)$.[10]. First, using $F(.)$ we have

$$\begin{aligned}
\frac{\pi_n}{\pi_{n-1}} &= \frac{P(n-1,n)}{P(n,n-1)} = \\
&= \frac{p_{s_2}(n-1)}{p_{s_1}(n)} \frac{\exp[\beta(\frac{n(s_1)-1}{2N-1}u_{11} + \frac{2N-1-(n(s_1)-1)}{2N-1}u_{12})]}{\exp[\beta(\frac{n(s_1)-1}{2N-1}u_{21} + \frac{2N-1-(n(s_1)-1)}{2N-1}u_{22})]} \\
&= \frac{p_{s_2}(n-1)}{p_{s_1}(n)} \frac{\exp[\beta(\frac{n(s_1)-1}{2N-1}(\frac{J}{2}+2h) + \frac{n(s_2)}{2N-1}(-\frac{J}{2}))]}{\exp[\beta(\frac{n(s_1)-1}{2N-1}(-\frac{J}{2}) + \frac{n(s_2)}{2N-1}(\frac{J}{2}-2h))]} \\
&= \frac{n(s_2)+1}{n(s_1)} \frac{\exp[\beta(\frac{n(s_1)-1}{2N-1}2h + \frac{n(s_1)-n(s_2)-1}{2N-1}\frac{J}{2})]}{\exp[\beta(\frac{n(s_1)-n(s_2)-1}{2N-1}(-\frac{J}{2}) - \frac{n(s_2)}{2N-1}(2h))]} \\
&= \frac{N-n+1}{N+n}\exp[\beta(\frac{n(s_1)-n(s_2)-1}{2N-1}J+2h)] \\
&= \frac{N-n+1}{N+n}\exp[\beta(\frac{2n-1}{2N-1}J+2h)] \\
&= \frac{N-n+1}{N+n}\exp[\beta(\frac{n}{2N-1}2J - \frac{1}{2N-1}J+2h)] \\
&= \frac{N-n+1}{N+n}\exp[\beta(2n\tilde{J} - \tilde{J}+2h)]
\end{aligned}$$

where $\tilde{J} = \frac{1}{2N-1}J$. Now, using Theorem 1, we can derive π_n for each n with $1 \leq n \leq N$.[11]

$$\begin{aligned}
\pi_n &= \pi_0 \prod_{i=1}^{n} \frac{N-i+1}{N+i}\exp[\beta(2i\tilde{J} - \tilde{J}+2h)] = \\
&= \pi_0 \frac{N\cdot(N-1)\cdots(N-(n-1))}{(N+1)\cdot(N+2)\cdots(N+n)}\exp[\beta(\sum_{i=1}^{n} 2i\tilde{J} - n\tilde{J}+2hn)] \\
&= \pi_0 \frac{N!/(N-n)!}{(N+n)!/N!}\exp[\beta((\frac{1}{2}n^2+\frac{1}{2}n)2\tilde{J} - n\tilde{J}+2hn)] \\
&= \pi_0 \exp[\beta(n^2\tilde{J}+2hn) + \ln(\frac{(N!)^2}{(N+n)!(N-n)!})]
\end{aligned}$$

We now have the following result.

Proposition 1 *For N sufficiently large, the most likely state is at n^*, with:*
- $n^* > 0$ *if* $h > 0$
- $n^* < 0$ *if* $h < 0$

Proof.

Consider the function $f : [-N+1, N] \to R$ given by

$$f(x) = \frac{N-x+1}{N+x}\exp\left[\beta\left(\frac{2x-1}{2N-1}J+2h\right)\right]$$

[10]The case of uniform, global matching corresponds to a two-state model with a homogenous population which is well-known in the statistical mechanics literature (e.g. Weidlich (1991)).

[11]The case of $-N \leq n \leq -1$ is completely analogous and thus omittted.

Note that

$$\frac{df(x)}{dx} = \exp\left[\beta\left(\frac{2x-1}{2N-1}J+2h\right)\right] \frac{1}{(N+x)^2(2N-1)}\frac{1}{2\beta J}\left(-x^2+x+\left(1-\frac{2}{\beta J}\right)N^2+N+\frac{1}{2\beta J}\right)$$

To determine the sign of $\frac{df(x)}{dx}$, it is sufficient to determine the sign of

$$-x^2+x+\left(1-\frac{2}{\beta J}\right)N^2+N+\frac{1}{2\beta J}$$

which we denote by $h(x)$. Also, denote by $b := \frac{1}{2}\beta J$.
We consider 3 cases.
<u>**Case 1)**</u> $b<1$. Observe that h has a maximum at $x^* = \frac{1}{2}$, which corresponds to

$$h(\frac{1}{2}) = \frac{1}{4}+\left(1-\frac{2}{\beta J}\right)N^2+N+\frac{1}{2\beta J}$$

Since $1-\frac{2}{\beta J}<0$, it follows that, for large N:

$$h(x) \leq h\left(\frac{1}{2}\right) < 0$$

for all $x \in [-N+1, N]$. This implies that f decreases from

$$f(-N+1) = 2N\exp\left[\beta\left(-J+2h\right)\right] > 1$$

to

$$f(N) = \frac{1}{2N}\exp\left[\beta\left(J+2h\right)\right] < 1$$

We conclude that π_n has a unique maximum at n^*, with $f(n^*)=1$. Moreover, we have:

a) if $h>0$, then $n^*>0$, since $f(0) = \frac{N+1}{N}\exp\left[\beta\left(-\frac{1}{2N-1}J+2h\right)\right] > 1$ for large N.

b) if $h<0$, then $n^*<0$, since $f(0) = \frac{N+1}{N}\exp\left[\beta\left(-\frac{1}{2N-1}J+2h\right)\right] < 1$ for large N.

<u>**Case 2)**</u> $b=1$. In this case

$$h(x) = -x^2+x+N+\frac{1}{4}$$

The equation $h(x)=0$ has two real and distinct solutions, which we denote by x_1 and x_2, given by:

$$x_1 = \frac{1-\sqrt{\Delta}}{2} \text{ and } x_2 = \frac{1+\sqrt{\Delta}}{2}$$

where $\Delta = 1+4\left(N+\frac{1}{4}\right)$. Note that $-N+1<x_1<0<x_2<N$. We summarize the behavior of f in Table 1.

As $N\to\infty$, $f(x_1)\to\exp(\beta 2h)$ and $f(x_2)\to\exp(\beta 2h)$.

a) If $h>0$, then, for large N, both $f(x_1)>1$ and $f(x_2)>1$, which together with $f(-N+1)>1$ and $f(N)<1$ imply that π_n has a unique max at n^*, where $n^*>0$.

Table 1: Behavior of f in Case 2

x	$-N+1$............x_1............x_2........N
$f(x)$	$f(-N+1) \searrow f(x_1) \nearrow f(x_2) \searrow f(N)$

Table 2: Behavior of f in Case 3

x	$-N+1$............x_1............x_2........N
$f(x)$	$f(-N+1) \searrow f(x_1) \nearrow f(x_2) \searrow f(N)$

b) If $h < 0$, then, for large N, both $f(x_1) < 1$ and $f(x_2) < 1$, which together with $f(-N+1) > 1$ and $f(N) < 1$ imply that π_n has a unique max at n^*, where $n^* < 0$.
Case 3) $b > 1$. The equation $h(x) = 0$ has two real and distinct solutions, which we denote by x_1 and x_2 given by:

$$x_1 = \frac{1-\sqrt{\Delta}}{2} \text{ and } x_2 = \frac{1+\sqrt{\Delta}}{2}$$

where

$$\Delta = 1 + 4\left[\left(1 - \frac{2}{\beta J}\right) N^2 + N + \frac{1}{2\beta J}\right]$$

We summarize the behavior of f in Table 2.

As $N \to \infty$,

$$f(x_1) \to \frac{1+\sqrt{1-\frac{2}{\beta J}}}{1-\sqrt{1-\frac{2}{\beta J}}} \exp\left[\beta\left(-\sqrt{1-\frac{2}{\beta J}}J + 2h\right)\right]$$

and

$$f(x_2) \to \frac{1-\sqrt{1-\frac{2}{\beta J}}}{1+\sqrt{1-\frac{2}{\beta J}}} \exp\left[\beta\left(\sqrt{1-\frac{2}{\beta J}}J + 2h\right)\right]$$

Let's denote by

$$a_0 := \frac{1}{2}\left[\beta J\sqrt{1-\frac{2}{\beta J}} - \log\frac{1+\sqrt{1-\frac{2}{\beta J}}}{1-\sqrt{1-\frac{2}{\beta J}}}\right]$$

and by

$$a := \beta h$$

Observe that $f(x_1) < 1$ is equivalent to $a < a_0$, and $f(x_2) < 1$ is equivalent to $a < -a_0$. Depending on the values of a, we split our analysis into 3 subcases:
Subcase 3A). If $|a| < a_0$ then, for large N, $f(x_1) < 1$ and $f(x_2) > 1$, which together with $f(-N+1) > 1$ and $f(N) < 1$ imply that π_n has two maxima at n_1^* and n_2^*, with $n_1^* < x_1 < 0 < x_2 < n_2^*$, and one minimum at n_3^*, with $x_1 < n_3^* < x_2$. To determine the global maximum, we consider 2 subcases:

1) $h > 0$. Note that

$$f(0) = \frac{N+1}{N} \exp\left[\beta\left(\frac{-1}{2N-1}J + 2h\right)\right] \to \exp(2\beta h) > 1$$

Table 3: Extrema of π_n for different parameter values

	$b<1$	$b=1$	$b>1$
$\lvert a\rvert < a_o$	1 maximum	1 maximum	2 maxima, 1 minimum
$\lvert a\rvert = a_o$	1 maximum	1 maximum	2 maxima, 1 minimum
$\lvert a\rvert > a_o$	1 maximum	1 maximum	1 maximum

so the minimum is at $n_3^* < 0$. Using this fact, we can write:

$$\begin{aligned}\pi_{n_2^*} &\geq \pi_{-n_1^*} = \pi_0 \frac{(N!)^2}{(N+n_1^*)!\,(N-n_1^*)!} \exp\left[\beta\left(\left(n_1^*\right)^2 \tilde{J} - 2hn_1^*\right)\right] \\ &= \pi_{n_1^*} \exp\left(-\beta 4hn_1^*\right) > \pi_{n_1^*}\end{aligned}$$

so the global max is at $n_2^* > 0$.

2) $h < 0$. Note that

$$f(0) = \frac{N+1}{N} \exp\left[\beta\left(\frac{-1}{2N-1}J + 2h\right)\right] \to \exp(2\beta h) < 1$$

so the min is at $n_3^* > 0$. Using this fact, we can write:

$$\begin{aligned}\pi_{n_1^*} &\geq \pi_{-n_2^*} = \pi_0 \frac{(N!)^2}{(N+n_2^*)!\,(N-n_2^*)!} \exp\left[\beta\left(\left(n_2^*\right)^2 \tilde{J} - 2hn_2^*\right)\right] \\ &= \pi_{n_2^*} \exp\left(-\beta 4hn_2^*\right) > \pi_{n_2^*}\end{aligned}$$

so the global max is at $n_1^* < 0$.

Subcase 3B) If $|a| > a_0$ then, for large N, $f(x_1) > 1$, $f(x_2) > 1$ or $f(x_1) < 1$, $f(x_2) < 1$. In both cases, π_n has a unique max at n^*, with $f(n^*) = 1$. Note that:

- if $a > a_0$, that is $h > 0$, the global maximum is at $n^* > 0$
- if $a < -a_0$, that is $h < 0$, the global maximum is at $n^* < 0$.

Subcase 3C) Finally $|a| = a_0$. First consider $a = a_0$. Then $f(x_1) \to 1$ and $f(_2) > 1$ (for large N). We can show that $f(x_1)$ is increasing in N, and since $f(x_1) \to 1$, we necessarily have $f(x_1) < 1$, for sufficiently large N. These observations, together with $f(-N+1) > 1$ and $f(N) < 1$, imply that π_n has two maxima and one minimum.

The second case $a = -a_0$ is similar i.e. π_n has two maxima and one minimum.

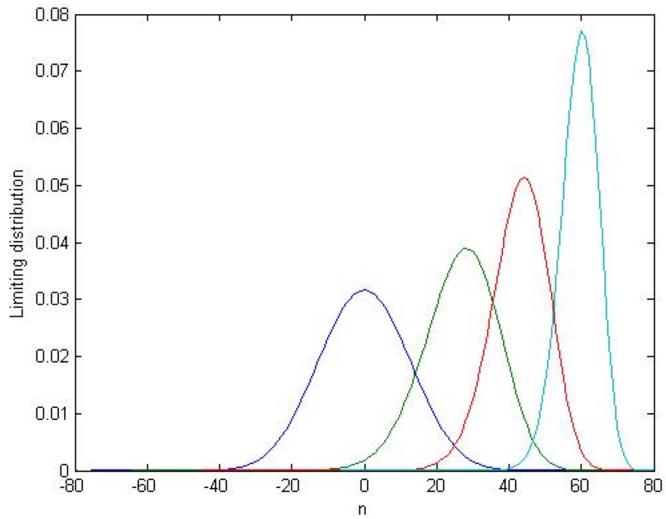
Moreover, we can prove the following (by analogy with subcase 3A):

- if $a = a_0$, that is $h > 0$, the global maximum is at $n^* > 0$
- if $a = -a_0$, that is $h < 0$, the global maximum is at $n^* < 0$. **QED**

We summarize the results in Table 3.

Of course, we can also derive the stochastically stable states for coordination games. Assume first that $h > 0$. As $\beta \to \infty$, we have $b \to \infty$ and $a < a_0$ (actually we can show that $(a - a_0) \to -\infty$) so we are in case 3A from above. But in this case we know that the global maximum is at n_2^*, with $x_2 < n_2^* < N$. Since, as $\beta \to \infty$, $f(x_2) \to \infty$ and $f(N) \to 1$ from below, it must be that $n_2^* \to N$. The case $h < 0$ is similar, i.e. the global maximum $n_1^* \to -N$. That is, in the limit, all players are expected to coordinate on the risk-dominant equilibrium.

Figure 1: As a increases, the mode shifts towards $N = 80$



However, by using Table 3 we can also identify the qualitative features of the limiting distribution for substantial noise. In the case of $b = \frac{1}{2}\beta J < 1$, the limiting distribution is unimodal and coordination on one of the pure Nash-equilibria does not occur. For $a = 0$ the mode is at state $n = 0$, but shifts towards $n = N$ for increasing a i.e. eveybody coordinating on s_1. This is illustrated in Figure 1. We note that for $a = 0$, the distribution has a peak at $n = 0$, but as we increase a (by increasing h) the peak shifts towards $n = 80$. Also we note that for high values of a, most of the distribution mass is concentrated around the mode. That is, most of the players are expected to play the risk-dominant strategy.

At $b = \frac{1}{2}\beta J > 1$, on the other hand, the limiting distribution is bimodal with both modes close to the extremes. The most likely states are those where almost all players play the same action. We refer to this phenomenon as *spontaneous coordination*. Note that the global maximum depends on a (more specifically depends on h) and is close to the risk-dominant equilibrium. This is illustrated in Figure 2. The plotted distribution is bi-modal, with the highest peak close to $N = 80$. We note that for high values of J (benefit from coordinating), most of the distribution mass is concentrated around the highest peak, so the players will coordinate on the risk-dominant strategy, which in this example is s_1.

Coordination is thus not a limit phenomenon, but may emerge even if individual choice is characterized by substantial noise. For high noise levels (β close to 0) spontaneous coordination will emerge for sufficiently high payoff differences between the coordinated and miscoordinated case captured by parameter J. This can be clearly seen in Figure 3: even though β is close to 0 (in our example we set $\beta = 1.7$), increasing J determines a shift of the highest peak towards the risk-dominant strategy. The issue of risk dominance captured by the parameter h is of secondary importance. It only determines which state is more likely on average. This insight is hidden in the double-limit analyses, since an increase in β simultaneously changes both parameters. One could imagine to test this model using experimental data. Then our model would predict that for low levels of J coordination will not be present, but will suddenly emerge as we increase the benefits from coordinating.

Figure 2: Highest peak is close to $N = 80$

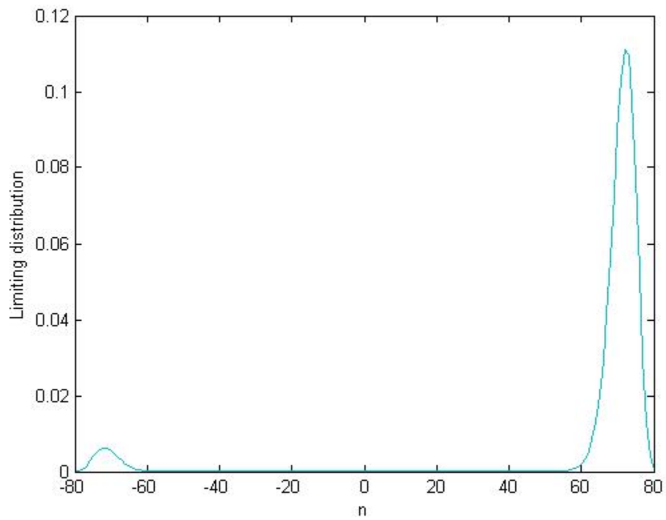
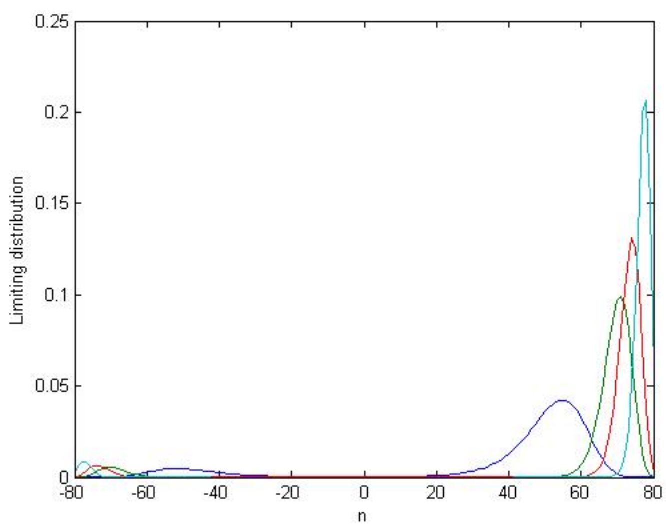


Figure 3: Increasing J shifts the mode towards the risk-dominant strategy



CONCLUSION

In this paper we study large potential games with global random matching. This approach allows us to analyze coordination even if individual choice behavior exhibits substantial levels of noise. The usual selection results (e.g. Blume (1993)) are derived as a corollary. For the case of many actors we derive a simple closed form representation of the unique stationary distribution. We show that coordination is not a limited phenomenon, but may occur even for substantial noise levels, depending on the relative benefits from coordinating as specified by the payoff matrix.

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